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Distillability and PPT entanglement of low-rank quantum states

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The bipartite quantum states ρ, with rank strictly smaller than the maximum of the ranks of the reduced states ρ_A and ρ_B, are distillable by local operations and classical communication [37]. Our first main result is that this is also true for NPT states with rank equal to this maximum. (A state is PPT if the partial transpose of its density matrix is positive semidefinite, and otherwise it is NPT.) This was conjectured first in 1999 in the special case when the ranks of ρ_A and ρ_B are equal (see [37, arXiv preprint]). Our second main result provides a complete solution of the separability problem for bipartite states of rank 4. Namely, we show that such a state is separable if and only if it is PPT and its range contains at least one product state. We also prove that the so called checkerboard states are distillable if and only if they are NPT.

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I. INTRODUCTION

Pure quantum entanglement plays the essential role in various quantum information tasks, such as GHZ states in quantum teleportation [5] and graph states in quantum computation [11]. However pure entangled states are always coupled with the environment due to the unavoidable decoherence. As a result they become mixed states which usually cannot be directly used, or become quite useless for quantum information tasks [46].

Hence, extracting pure entanglement from mixed states is a basic task in quantum information. Formally, this task is called *entanglement distillation* and the entanglement measure quantifying the asymptotically obtainable pure entanglement is called *distillable entanglement*; the states from which we can obtain pure entanglement are *distillable* [9]. Apart from the mentioned physical applications, the distillable entanglement is the lower bound of many well-known entanglement measures such as the entanglement cost, entanglement of formation [9] and the squashed entanglement [7, 18]. An upper bound for distillable entanglement is provided by the distillable key [17, 35], and recently the gap between them has been experimentally observed [20]. A lot of effort has been devoted to the investigation of entanglement distillation, for a review see [38].

For a state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ the partial transpose, computed in an orthonormal (o.n.) basis $\{|a_i\rangle\}$ of system A, is defined by $\rho^\Gamma := \sum_{ij} |a_i\rangle\langle a_j| \otimes \langle a_j|\rho|a_i\rangle$. Mathematically, we say that ρ is *1-distillable* if there exists a pure bipartite state $|\psi\rangle$ of Schmidt rank 2 such that $\langle\psi|\rho^\Gamma|\psi\rangle < 0$ [21]. More generally, we say that ρ is *n-distillable* if the state $\rho^{\otimes n}$ is 1-distillable. Finally, we say that ρ is *distillable* under local operations and classical communication (LOCC) if it is n-distillable for some $n \geq 1$.

It is a famous open question whether all bipartite NPT states, i.e., the states ρ such that ρ^Γ has at least one negative eigenvalue, are distillable. Although many papers deal with the problem of distillation of pure entanglement from NPT states [3, 15, 38, 44, 50], the complete solution is still unknown. In this paper we will solve an open question, which is an important special case of the distillation problem.

It follows easily from the definition of distillability given above that bipartite PPT states, i.e., states with positive semidefinite partial transpose, are not distillable by LOCC [30, 32]. In recent years, PPT entangled states have been extensively studied in connection with the phenomena of entanglement activation and universal usefulness [33, 42], the distillable key [35], the symmetry permutations [49] and entanglement witnesses [40], both in theory and experiment [1].

Therefore, it is an important and basic question to decide whether a given PPT state is separable [38]. The famous partial transpose criterion says that the separable states are PPT [45]. Furthermore Horodeckis showed that this is necessary and sufficient for states in $M \otimes N$ quantum systems with $MN \leq 6$ [29]. However the problem becomes difficult for any bigger dimension. For instance, the first examples of PPT entangled states acting on $2 \otimes 4$ and $3 \otimes 3$

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were discovered in 1997 [30], but it is still an open problem to decide which PPT states in these systems are separable. The situation is not much better if we consider the separability problem for bipartite states of fixed rank, where the cases of ranks 4 and 5 remained unresolved. We have discovered a simple criterion for separability of states of rank 4.

Throughout the paper the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional. We will ignore the normalization condition for states since it does not affect the process of entanglement distillation. Thus, unless stated otherwise, we assume that the states are non-normalized, and we set $M = \mathrm{Dim}\,\mathcal{H}_A$ and $N = \mathrm{Dim}\,\mathcal{H}_B$. As our definition of an "$M \times N$ state" can be easily misinterpreted, we state it formally.

Definition 1 *A bipartite state ρ is an $M \times N$* state *if the reduced states $\rho_A = \mathrm{Tr}_B(\rho)$ and $\rho_B = \mathrm{Tr}_A(\rho)$ have ranks M and N, respectively.*

Our first main result, Theorem 10, asserts that $M \times N$ rank-N NPT states are distillable under LOCC. The stronger statement that these states are in fact 1-distillable under LOCC is an immediate consequence of the proof. The special case $M = N$ was proposed in 1999 as a conjecture [37, Conjecture 1]. As pointed out in [37], it follows from this special case that all rank 3 entangled states are distillable. Hence we recover the main result of [14]. We give the details in Sec. II and III.

Our second main result, Theorem 22, solves the separability problem for bipartite states of rank 4. Namely, it asserts that a bipartite state of rank 4 is separable if and only if it is PPT and its range contains at least one product state. This is a numerically operational criterion, e.g., see the method introduced in [36]. Therefore, 3×3 PPT states of rank 4 constitute the first class of PPT states of fixed dimension and rank for which the separability problem can be operationally decided. We also discuss the analytical method based on the Plücker coordinates and the Grassmann variety. We give the details in Sec. IV, V, and the appendix.

We also apply our results to characterize the quantum correlations inside a tripartite pure state $|\psi\rangle$. In Sec. VI, Theorem25, we show that the reduced density operators ρ_{AB} and ρ_{AC} of $\rho = |\psi\rangle\langle\psi|$ are undistillable if and only if they are separable. In other words, there is no tripartite pure state with two undistillable entangled reduced density operators. We also discuss the relation between distillability and quantum discord [13], where the latter is another kind of quantum correlation.

To investigate the states beyond $M \times N$ states of rank N, we study the family of checkerboard states. This family consists of two-qutrit states [23], generically of rank 4, which generalizes the family constructed by Bruß and Peres [12]. In Sec. VII we prove that the NPT checkerboard states are 1-distillable. In Sec. VIII we analyze further the full-rank properties (defined below) used in the proof of our first main result. We conclude and discuss our results in Sec. IX.

For convenience and reference, we list some known results on distillability of bipartite NPT states under LOCC:

(A) The $2 \times N$ NPT states are distillable [6, 21, 31].

(B) The states violating the reduction criterion are distillable [28].

(C) The $M \times N$ states of rank less than M or N are distillable [37].

(D) The NPT states of rank at most three are distillable [14].

(E) All NPT states can be converted by LOCC into NPT Werner states [19, 21, 52].

The states mentioned in (A-D) are in fact 1-distillable. For the states in result (A), it suffices to observe that generic pure states in a $2 \otimes N$ space have Schmidt rank 2. For (B) see [15] or the next section. The 1-distillability in result (C) follows from that of (B) and the proof of [37, Theorem 1]. The assertion for states in result (D) can be deduced from those of (A) and (B) [14].

In view of the result (E), one might hope to solve the problem of entanglement distillation by distilling the NPT Werner states. Unfortunately, although this problem has been studied extensively in the past decade, it still remains an open and apparently very hard problem [44]. It is thus of high importance to address the non-Werner states and propose useful tools for their distillation. In particular, one can see that the result (B), namely reduction criterion, plays quite important role for distillation of generic entangled states. For example, we have argued that other three main results (A), (C), (D) are partially or totally derivable from the reduction criterion.

Let us introduce the right full-rank property via the Hermitian observable $\mathrm{Tr}_A(|x\rangle\langle x|\rho) = \langle x|\rho|x\rangle$ where $|x\rangle \in \mathcal{H}_A$. We say that ρ has the *right full-rank property (RFRP)* if the operator $\langle x|\rho|x\rangle$ is invertible (i.e., has full rank) for some $|x\rangle \in \mathcal{H}_A$. One defines similarly the *left full-rank property (LFRP)* by using the Hermitian observable $\mathrm{Tr}_B(|y\rangle\langle y|\rho) = \langle y|\rho|y\rangle$ with $|y\rangle \in \mathcal{H}_B$. Every result about RFRP has its analog for LFRP, and we shall work mostly with the former.

We point out that RFRP has appeared previously in [36] where it was shown that all $M \times N$ ($M \leq N$) PPT states of rank N have RFRP (see the proof of Theorem 1 in that paper).

We prove in Theorem 8 that bipartite states violating LFRP or RFRP are distillable under LOCC. We refer to this result as the *full-rank criterion*. It is a crucial ingredient in the proof of Theorem 10. Let us also mention that Theorems 8 and 10 remain valid if we replace the word "distillable" with "1-distillable" (see the next section).

The questions of separability and distillability for the $M \times N$ states ρ of rank $R \leq N$, with $M \leq N$, have very simple answer: they are separable if and only if they are PPT, and they are distillable if and only if they are NPT. The result on separability is proved in [36], and the one on distillability in [37, Theorem 1] when $R < N$, while our first main result handles the case $R = N$.

As a transition between the first and second main result, we introduce the concepts of reducible and irreducible quantum states. The sum ρ of bipartite states ρ_i is *B-direct* if $\mathcal{R}(\rho_B)$ is a direct sum of $\mathcal{R}((\rho_i)_B)$. A bipartite state ρ is *reducible* if it is a B-direct sum of two states and otherwise ρ is *irreducible*. In the language of quantum state transformation, a reducible state is stochastic-LOCC (SLOCC) equivalent to the sum of ρ_i whose reduced density operators $(\rho_i)_B$ are pairwise orthogonal [22]. We formally state this fact in Proposition 15. More properties of reducible states are also introduced, e.g., to study the distillability in Proposition 17, 18 and PPT, separability in Corollary 16.

Based on the reducibility, we can easily decide the separability and distillability of the bipartite state ρ of rank 4 when there is some $|x\rangle \in \mathcal{H}_A$ such that the operator $\langle x|\rho|x\rangle$ is non-invertible. This is demonstrated in Lemma 20. We solve the same problem when there is a product state in the range of ρ in Proposition 21. In terms of these results we reach our second main result. That is, a bipartite state of rank 4 is separable if and only if it is PPT and there is a product state in its range. This is illustrated in Theorem 22 and it is numerically operational. We also propose an analytical method in terms of the Grassmann variety to implement our result. In the case of 2-dimensional subspaces of a $2 \otimes 3$ space we give a simple equation which is satisfied if and only if the subspace contains a product state. Similarly, in the appendix we give a single equation test for the existence of a product state in the 3-dimensional subspace of a $2 \otimes 4$ space. Such equation also exists in the case of 4-dimensional subspaces of a $3 \otimes 3$ space but we were not able to compute it. The method is totally analytical and does not rely on numerical estimation. The reducibility also constitutes the technical bases for the distillability of checkerboard states, as proved in Theorem 28.

We also apply the main results to derive a few physical facts. For example, we prove in Theorem 25 a tripartite pure state cannot have two undistillable entangled reduced density operators. This is realized by the fact that the two reduced density operators have a maximal local rank equal to its rank, as well as the first main result and Proposition 3.

Throughout the paper we write I_k resp. 0_k for the identity resp. zero $k \times k$ matrix. The inequality $H \geq 0$ means that H is a positive semidefinite Hermitian operator or matrix. Similarly, $H > 0$ means that H is positive definite. We denote by $\mathcal{R}(\rho)$ the range space of an operator ρ.

We write $[X, Y]$ for the commutator $XY - YX$. We recall that if X is normal and Y arbitrary then $[X, Y] = 0$ implies $[X^\dagger, Y] = 0$. This follows from the fact that, for a normal operator X, there exists a polynomial $f(t)$ such that $X^\dagger = f(X)$.

II. PRELIMINARY FACTS

Before proceeding with the proofs of our results, let us recall some additional facts. First, by using an o.n. basis $\{|u_i\rangle\}$ of $\mathcal{H}_A$, we can write any state as $\rho = \sum_{i,j=1}^{M} |u_i\rangle\langle u_j| \otimes \sigma_{ij}$. Hence ρ is represented by its matrix

$$\rho = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1M} \\ \sigma_{21} & \sigma_{22} & & \sigma_{2M} \\ \vdots & & & \\ \sigma_{M1} & \sigma_{M2} & & \sigma_{MM} \end{pmatrix}. \tag{1}$$

Usually we assume that $\{|u_i\rangle\}$ is the computational basis $\{|i\rangle\}$.

Second, the RFRP (defined in the Introduction) was an essential tool used in [36] and will also play an important role in this paper. Clearly, RFRP is equivalent to the following assertion: One can choose an o.n. basis $\{|u_i\rangle\}$ of $\mathcal{H}_A$ such that some diagonal block σ_{ii} in the above matrix has rank N.

The condition that $\operatorname{rank}(\langle x|\rho|x\rangle) = N$ is equivalent to $\ker\langle x|\rho|x\rangle = 0$. It follows that ρ violates RFRP if and only if for each state $|x\rangle \in \mathcal{H}_A$ there exists a state $|y\rangle \in \mathcal{H}_B$ such that $|x\rangle \otimes |y\rangle \in \ker \rho$.

Example 2 We use an argument from the proof of [36, Lemma 3], to show that all $M \times N$ separable states ρ have RFRP. We can write ρ as a finite sum of non-normalized product states $\rho = \sum_i |a_i, b_i\rangle\langle a_i, b_i|$. Choose $|x\rangle \in \mathcal{H}_A$ such that $\langle x|a_i\rangle \neq 0$ for all i. As the $|b_i\rangle$ span $\mathcal{H}_B$, $\langle x|\rho|x\rangle$ has rank N.

We will show in the next section that this result extends to all PPT states.

We shall need the following result proved in [36].

Proposition 3 *If ρ is an $M\times N$ PPT state with $M\le N$ and rank N, then ρ is separable and can be written as a sum of N product states*

$$\rho=\sum_{i=1}^{N}|a_i,b_i\rangle\langle a_i,b_i|.$$

If A and B are linear operators on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, then we shall refer to $A\otimes B$ as a *local operator*. The abbreviation ILO will refer to invertible local operators, i.e., to operators $A\otimes B$ with both A and B invertible. It is easy to see that RFRP is invariant under ILOs, i.e., if $A\otimes B$ is an ILO and the state ρ has RFRP then the transformed state $(A\otimes B)^\dagger\ \rho\ (A\otimes B)$ also has RFRP. Moreover, if ρ violates RFRP then so does $(A\otimes B)^\dagger\ \rho\ (A\otimes B)$ even if $A\otimes B$ is not invertible.

Third, there is a simple way to prove distillability of ρ which can be applied in many cases. For that purpose observe that if ρ is given by its matrix (1) then

$$\rho^\Gamma=\begin{pmatrix}\sigma_{11} & \sigma_{21} & \cdots & \sigma_{M1}\\ \sigma_{12} & \sigma_{22} & & \sigma_{M2}\\ \vdots & & & \\ \sigma_{1M} & \sigma_{2M} & & \sigma_{MM}\end{pmatrix}.$$

Since ρ is a Hermitian matrix, so is ρ^Γ. Let $X=\begin{pmatrix} a & b\\ b^* & c\end{pmatrix}$ be a principal 2×2 submatrix of ρ^Γ. Thus a and c are diagonal entries of ρ^Γ.

Lemma 4 *If $ac=0$ while $b\ne 0$ then ρ is distillable.*

Proof. Since X has a negative eigenvalue and the diagonal blocks $\sigma_{ii}\ge 0$, the diagonal entries a and c must belong to different diagonal blocks, say σ_{kk} and σ_{ll}. Let P be the orthogonal projector onto the 2-dimensional subspace of $\mathcal{H}_A$ spanned by $|k\rangle$ and $|l\rangle$. Then the projected state $\rho':=(P\otimes I_N)\ \rho\ (P\otimes I_N)$ is an NPT state acting on a $2\otimes N$ system. Hence ρ' is distillable by the result (A). Consequently, ρ is also distillable. □

Whenever the distillability of ρ can be proved by direct application of this simple lemma, we shall say that the state ρ is *trivially distillable*. For example, by using this concept we generalize Theorem 2 of Ref. [37].

Theorem 5 *Let $\rho=|\psi\rangle\langle\psi|+\sigma$ be an $M\times N$ state where σ is any state and $|\psi\rangle$ is a pure entangled state. If $r:=\mathrm{rank}(\sigma_A)<M$, then ρ is distillable.*

Proof. We may assume that $\{|1\rangle,\dots,|r\rangle\}$ is a basis of $\mathcal{R}(\sigma_A)$. We can write $|\psi\rangle=\sum_{i=1}^M|i,\psi_i\rangle$ where $|\psi_i\rangle\in\mathcal{H}_B$. Since $r<M$, we must have $|\psi_M\rangle\ne 0$. Since $|\psi\rangle$ is entangled, at least one of the $|\psi_k\rangle$, $k<M$, is not parallel to $|\psi_M\rangle$. Let us fix such index k. Let V be an ILO on $\mathcal{H}_B$ such that $V|\psi_k\rangle=|1\rangle$ and $V|\psi_M\rangle=|2\rangle$. If $P=|k\rangle\langle k|+|M\rangle\langle M|$, then the state $\rho':=(P\otimes V)\ \rho\ (P\otimes V)^\dagger$ acts on a $2\otimes N$ system.

We claim that ρ' is trivially distillable. Since this is the first time that we are applying Lemma 4, we shall give all the details. Since $P\otimes V|\psi\rangle=|k,1\rangle+|M,2\rangle$, we have

$$\begin{aligned} P\otimes V\ |\psi\rangle\langle\psi|\ P\otimes V^\dagger &= |k,1\rangle\langle k,1|+|M,2\rangle\langle M,2|+|k,1\rangle\langle M,2|+|M,2\rangle\langle k,1|,\\ \left(P\otimes V\ |\psi\rangle\langle\psi|\ P\otimes V^\dagger\right)^\Gamma &= |k,1\rangle\langle k,1|+|M,2\rangle\langle M,2|+|M,1\rangle\langle k,2|+|k,2\rangle\langle M,1|.\end{aligned}$$

On the other hand since $\sigma\ (|M\rangle\langle M|\otimes V)^\dagger=0$, the $2N\times 2N$ matrix of $(P\otimes V)\ \sigma\ (P\otimes V)^\dagger$ has its nonzero entries all in the $N\times N$ submatrix contained in the first N rows and columns. Hence, the 2×2 principal submatrix of $(\rho')^\Gamma$ in rows 2 and $N+1$ has the form $\begin{pmatrix} * & 1\\ 1 & 0\end{pmatrix}$, which proves our claim. It follows from this claim that ρ is also distillable. □

Theorem 2 of Ref. [37] is obtained by taking σ to be a pure product state.

In the following proposition we represent $M\times N$ states in a convenient matrix form which will be used in our proofs. We include a result proved in [36] for the states which are also PPT.

Proposition 6 *Let ρ be an $M\times N$ state of rank R.*

(i) ρ can be written as

$$\rho \;=\; \sum_{i,j} |i\rangle\langle j| \otimes C_i^\dagger C_j = (C_1,\ldots,C_M)^\dagger \cdot (C_1,\ldots,C_M), \tag{2}$$

where C_i are $R\times N$ matrices such that $\rho_B = \sum_i C_i^\dagger C_i > 0$.

(ii) If $R = N$ and ρ has RFRP, then there exists an invertible local operator $A\otimes B$ such that

$$\begin{aligned}\rho' &:= (A\otimes B)^\dagger \;\rho\; A\otimes B \\ &= (C_1,\ldots,C_{M-1},I_N)^\dagger \cdot (C_1,\ldots,C_{M-1},I_N).\end{aligned}$$

Moreover, ρ' is PPT if and only if the C_i are pairwise commuting normal matrices.

Proof. The assertion (i) follows from the spectral decomposition theorem. Indeed, by that theorem we have

$$\rho = \sum_{i=1}^{R} |\psi_i\rangle\langle\psi_i|, \quad |\psi_i\rangle = \sum_{j=1}^{M} |j,\psi_{ij}\rangle, \tag{3}$$

where $|\psi_i\rangle$ are non-normalized pure states. Since these states span the range of ρ, they must be linearly independent. However, they are not uniquely determined by ρ. The matrices C_j can be chosen as follows:

$$C_j = (|\psi_{1j}\rangle,\ldots,|\psi_{Rj}\rangle)^\dagger\,, \quad j=1,\ldots,M. \tag{4}$$

In other words, $\langle\psi_{ij}|$ is the ith row of C_j. Since $\rho_B \ge 0$ and $\operatorname{rank}\rho_B = M$, we have $\rho_B > 0$.

For the first assertion of (ii) we may assume that C_M is invertible and then apply the local operator $I_A \otimes C_M^{-1}$. It is shown in [36] that the PPT condition implies that the C_is are pairwise commuting normal matrices. The converse is straightforward. $\square$

The next example shows that there exist $M\times N$ NPT states of rank N which violate RFRP.

Example 7 For the (non-normalized) 3×3 rank-3 antisymmetric state

$$\rho_{as} = \sum_{1\le i<j\le 3} (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|), \tag{5}$$

the operator $\langle x|\rho_{as}|x\rangle$ has rank 2 for all nonzero vectors $|x\rangle_A = \sum \xi_i|i\rangle \in \mathcal{H}_A$. This can be verified as follows. We first compute the matrices C_i. In this example we can choose $|\psi_i\rangle = |jk\rangle - |kj\rangle$ where (i,j,k) is a cyclic permutation of $(1,2,3)$. Then we find that

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

Thus $\langle x|\rho_{as}|x\rangle = X^\dagger X$ where $X = \sum_i \xi_i C_i$. Since X is antisymmetric its rank must be even, and so both X and $\langle x|\rho_{as}|x\rangle$ have rank 2.

For a more general class of examples violating RFRP see section VIII. The problem of characterizing the states violating RFRP remains open.

Let us make two important observations. First, there is a stronger version of the result (B) from the Introduction, due to Clarisse [15]. Namely, he has shown that a bipartite state ρ which violates the reduction criterion is in fact 1-distillable [21]. This is much stronger then being merely distillable.

Second, assume that a bipartite state σ is obtained from ρ by applying a local operator: $\sigma = (A\otimes B)^\dagger\ \rho\ (A\otimes B)$, where $A\otimes B$ may be singular. If σ is 1-distillable then ρ is also 1-distillable. Indeed if the Hermitian operator D is a 1-distillation witness detecting σ, i.e., $\mathrm{Tr}(\sigma D) < 0$ while $\mathrm{Tr}(\sigma' D) \ge 0$ for all 1-undistillable states σ', then $(A\otimes B)\ D\ (A\otimes B)^\dagger$ is a 1-distillation witness detecting ρ.

One can easily verify that in all cases that arise in our proofs, the above two observations garantee that the state ρ from which we start is not only distillable but also 1-distillable. Hence Theorems 8 and 10 remain valid when we replace the word "distillable" with "1-distillable".

III. DISITILLABILITY OF $M \times N$ NPT STATES OF RANK N

In this section we prove our main result Theorem 10. However, our first objective is to show that the states which violate LFRP or RFRP are distillable. Since PPT states are not distillable, it follows that all PPT states must possess both LFRP and RFRP.

For convenience, we shall denote by $X[k]$ the submatrix of the matrix X consisting of the last k columns. If a matrix X factorizes as $X = YZ$, then we shall say that Z is a *right factor* of X.

Theorem 8 *Bipartite states which violate LFRP or RFRP are distillable.*

Proof. It suffices to prove that any $M \times N$ state ρ which violates RFRP is distillable. Let R be the rank of ρ. If $R < N$, then ρ is distillable by result (C). From now on we assume that $R \geq N$. By Proposition 6 we have $\rho = (C_1, \ldots, C_M)^\dagger \cdot (C_1, \ldots, C_M)$ where each matrix C_i is of size $R \times N$, and $\rho_B = \sum_i C_i^\dagger C_i > 0$.

We can replace $(C_1, \ldots, C_M)$ with $U(C_1, \ldots, C_M)$ where U is a unitary matrix, without changing ρ. The effect of an invertible local transformation $\rho \to (I \otimes B)^\dagger \, \rho \, (I \otimes B)$ is to replace each C_i by $C_i B$. Recall that RFRP is preserved by these local transformations. In order to prove the theorem we can apply these kind of transformations as many times as needed.

Since ρ_A is invertible, each $C_i \neq 0$. Let r_1 be the rank of C_1. By multiplying $(C_1, \ldots, C_M)$ by a unitary matrix U_1 on the left hand side, we may assume that the last $R - r_1$ rows of C_1 are zero. We choose an invertible matrix B_1 such that $C_1 B_1 = I_{r_1} \oplus 0$. By multiplying all C_is by B_1 on the right hand side, we may assume that $C_1 = I_{r_1} \oplus 0$. Since ρ violates RFRP, we have $r_1 < N$.

For a sequence of indexes $1 \leq i_1 < i_2 < \cdots < i_k \leq M$ we denote by $\rho_{i_1,\ldots,i_k}$ the corresponding principal submatrix of ρ of size $kN \times kN$, i.e.,

$$\rho_{i_1,\ldots,i_k} = (C_{i_1}, \ldots, C_{i_k})^\dagger \cdot (C_{i_1}, \ldots, C_{i_k}).$$

The corresponding principal submatrix of ρ^Γ will be denoted by $\rho^\Gamma_{i_1,\ldots,i_k}$. For instance, for $i > 1$ we have

$$\rho_{1,i} = (C_1, C_i)^\dagger \cdot (C_1, C_i) = \begin{pmatrix} C_1^\dagger C_1 & C_1^\dagger C_i \\ C_i^\dagger C_1 & C_i^\dagger C_i \end{pmatrix}$$

and

$$\rho^\Gamma_{1,i} = \begin{pmatrix} C_1^\dagger C_1 & C_i^\dagger C_1 \\ C_1^\dagger C_i & C_i^\dagger C_i \end{pmatrix}.$$

We split each C_i into four blocks $C_i = \begin{pmatrix} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{pmatrix}$ with C_{i1} square of size r_1. Since $C_1 = I_{r_1} \oplus 0$, we have

$$\rho_{1,i} = \begin{pmatrix} I_{r_1} & 0 & C_{i1} & C_{i2} \\ 0 & 0 & 0 & 0 \\ C_{i1}^\dagger & 0 & * & * \\ C_{i2}^\dagger & 0 & * & * \end{pmatrix}, \quad i > 1,$$

where the asterisk stands for an unspecified block. If some $C_{i2} \neq 0$, then ρ is trivially distillable. Thus we may assume that all $C_{i2} = 0$.

Since ρ_B has rank N, its submatrix $\rho_B[N - r_1]$ must have rank $N - r_1$. Since $C_{i2} = 0$, C_{i4} is a right factor of the submatrix $(C_i^\dagger C_i)[N - r_1]$. Therefore $C_{i4} \neq 0$ for at least one index $i > 1$. By permuting the C_is with $i > 1$, we may assume that $C_{24} \neq 0$. Let r_2 be its rank. By multiplying $(C_1, \ldots, C_M)$ by a unitary matrix $I_{r_1} \oplus U_2$ on the left hand side, we may assume that the last $R - r_1 - r_2$ rows of C_{24} are zero. Let B_2 be an invertible matrix such that $C_{24} B_2 = I_{r_2} \oplus 0$. By multiplying each C_i by $I_{r_1} \oplus B_2$ on the right hand side, we may assume that $C_{24} = I_{r_2} \oplus 0$. Note that these operations performed on $(C_1, \ldots, C_M)$ do not alter C_1 and that the equalities $C_{i2} = 0$ remain valid.

Assume that $r_1 + r_2 < N$. We split each C_{i4} into four blocks $C_{i4} = \begin{pmatrix} C_{i41} & C_{i42} \\ C_{i43} & C_{i44} \end{pmatrix}$, with C_{i41} square of size r_2. For $i > 2$ we have

$$\rho_{2,i} = \begin{pmatrix} C_2^\dagger C_2 & C_2^\dagger C_i \\ C_i^\dagger C_2 & C_i^\dagger C_i \end{pmatrix} = \begin{pmatrix} * & C_{23}^\dagger C_{24} & * & C_{23}^\dagger C_{i4} \\ C_{24}^\dagger C_{23} & C_{24}^\dagger C_{24} & C_{24}^\dagger C_{i3} & C_{24}^\dagger C_{i4} \\ * & C_{i3}^\dagger C_{24} & * & C_{i3}^\dagger C_{i4} \\ C_{i4}^\dagger C_{23} & C_{i4}^\dagger C_{24} & C_{i4}^\dagger C_{i3} & C_{i4}^\dagger C_{i4} \end{pmatrix}.$$

We extract from it the principal submatrix

$$\begin{pmatrix} C_{24}^\dagger C_{24} & C_{24}^\dagger C_{i4} \\ C_{i4}^\dagger C_{24} & C_{i4}^\dagger C_{i4} \end{pmatrix} = \begin{pmatrix} I_{r_2} & 0 & C_{i41} & C_{i42} \\ 0 & 0 & 0 & 0 \\ C_{i41}^\dagger & 0 & * & * \\ C_{i42}^\dagger & 0 & * & * \end{pmatrix}.$$

If some $C_{i42} \neq 0$, then ρ is trivially distillable. Thus we may assume that all $C_{i42} = 0$.

Since ρ_B has rank N, the matrix $\rho_B[N - r_1 - r_2]$ must have rank $N - r_1 - r_2$. Since $C_{i2} = 0$ and $C_{i42} = 0$, we have

$$C_i = \begin{pmatrix} C_{i1} & 0 & 0 \\ * & C_{i41} & 0 \\ * & C_{i43} & C_{i44} \end{pmatrix}.$$

Consequently, C_{i44} is a right factor of the submatrix $(C_i^\dagger C_i)[N - r_1 - r_2]$. Therefore $C_{i44} \neq 0$ for at least one index $i > 2$. By permuting the C_is with $i > 2$, we may assume that $C_{344} \neq 0$. Let r_3 be its rank. By multiplying $(C_1, C_2, \ldots, C_M)$ by a unitary matrix $I_{r_1+r_2} \oplus U_3$ on the left hand side , we may assume that the last $R - r_1 - r_2 - r_3$ rows of C_{344} are zero. Let B_3 be an invertible matrix such that $C_{344} B_3 = I_{r_3} \oplus 0$. By multiplying each C_i by $I_{r_1+r_2} \oplus B_3$ on the right hand side, we may assume that $C_{344} = I_{r_3} \oplus 0$. Note that these operations performed on $(C_1, C_2, \ldots, C_M)$ alter neither C_1 nor C_{24} and that the equalities $C_{i2} = 0$ and $C_{i42} = 0$ remain valid.

If $r_1 + r_2 + r_3 < N$ we can continue by splitting each C_{i44} into four blocks, etc. This process terminates as soon as ρ has been shown to be trivially distillable. Otherwise it can be continued as long as $r_1 + \cdots + r_k < N$ and $k < M$. However, note that if k becomes equal to M then we must have $r_1 + \cdots + r_M = N$ because $\operatorname{rank} \rho_B = N$.

We claim that the process must terminate while the inequality $r_1 + \cdots + r_k < N$ is still valid. Indeed assume that we reach a point where $r_1 + \cdots + r_k = N$. Since $k \leq M$ we can set $|x\rangle = t_1|1\rangle + \cdots + t_k|k\rangle \in \mathcal{H}_A$, where $t_1, \ldots, t_k$ are real parameters. Then we have

$$\langle x|\rho|x\rangle = \sum_{i,j=1}^{k} t_i t_j C_i^\dagger C_j = \left(\sum_{i=1}^{k} t_i C_i\right)^\dagger \left(\sum_{i=1}^{k} t_i C_i\right).$$

It follows that

$$\det\langle x|\rho|x\rangle = \left|\det\left(\sum_{i=1}^{k} t_i C_i\right)\right|^2.$$

The determinant on the left hand side is a polynomial in the parameters $t_1, \ldots, t_k$. As $C_{i2} = 0$, $C_{i42} = 0, \ldots$ for all i, the coefficient of $t_1^{2r_1} t_2^{2r_2} \cdots t_k^{2r_k}$ in this determinant is equal to 1. Hence, we can choose the values of the parameters t_i so that the operator $\langle x|\rho|x\rangle$ becomes invertible. This contradicts the hypothesis that ρ violates RFRP, and completes the proof. $\square$

Let us also record the observation we made earlier.

Corollary 9 *All bipartite PPT states possess LFRP and RFRP.*

For $M \times N$ states with $M \leq N$ and rank N this was proved in [36].

By using Theorem 8 we can now prove our main result.

Theorem 10 *The $M \times N$ NPT states of rank N are distillable under LOCC.*

Proof. By the result (C) we may assume that $M \leq N$. In view of Theorem 8, it suffices to prove the assertion for $M \times N$ NPT states ρ of rank N which have RFRP. Since ρ is NPT, we must have $M \geq 2$. By Proposition 6 we can assume that $\rho = (C_1, \ldots, C_{M-1}, I_N)^\dagger \cdot (C_1, \ldots, C_{M-1}, I_N)$ where C_i are $N \times N$ matrices. Let $\rho_i = (P_i \otimes I)\ \rho\ (P_i \otimes I)$ where $P_i = |i\rangle\langle i| + |M\rangle\langle M|$, $i < M$. If some ρ_i is NPT, then ρ_i (and ρ) is distillable by the result (A). Thus we may assume that all ρ_i are PPT. By applying Proposition 6 (ii) to ρ_i, we deduce that C_i is a normal matrix. Since ρ is NPT, the same proposition implies that there exist i, j such that $[C_i, C_j] \neq 0$. In particular, $M \geq 3$. We may assume that $i = 1$ and $j = 2$.

Let $\rho' = (V \otimes I_N)^\dagger\ \rho\ (V \otimes I_N)$ where $V = (x|1\rangle + |2\rangle)\langle 2| + |M\rangle\langle M|$ and x is a complex parameter. Obviously we have

$$(C_1, \ldots, C_{M-1}, I_N)(V \otimes I_N) = (0, X, 0, \ldots, 0, I_N),$$

where $X = xC_1 + C_2$. Since the range of ρ'_A is contained in the subspace spanned by $|2\rangle_A$ and $|M\rangle_A$, we can view ρ' as a state acting on a $2 \otimes N$ space. Then its density matrix is $\rho' = (X, I_N)^\dagger \cdot (X, I_N)$ and we have

$$\rho'_A = \begin{pmatrix} \mathrm{Tr}(X^\dagger X) & \mathrm{Tr}(X^\dagger) \\ \mathrm{Tr}(X) & N \end{pmatrix}.$$

Hence $\det(\rho'_A) = N\,\mathrm{Tr}(X^\dagger X) - |\,\mathrm{Tr}(X)|^2$. Since $[C_1, C_2] \neq 0$, the matrices X and I_N are linearly independent. Consequently, $\det(\rho'_A) > 0$ by the Cauchy–Schwarz inequality. Since $\rho'_B = I_N + X^\dagger X > 0$, we conclude that ρ' is a $2 \times N$ state for all x. Evidently its rank is N.

Assume that ρ' is PPT for all values of x. By Proposition 6, the matrix $X = xC_1 + C_2$ is normal. Since x is arbitrary and both C_1 and C_2 are normal, we deduce that $x[C_1, C_2^\dagger] = x^*[C_1^\dagger, C_2]$. By setting first $x = 1$ and then that x is the imaginary unit, we deduce that $[C_1, C_2^\dagger] = 0$. Consequently, also $[C_1, C_2] = 0$ because C_2 is normal, and we have a contradiction. We conclude that ρ' must be NPT for at least one value of x. Such ρ' is distillable by the result (A) and, consequently, ρ is also distillable. This completes the proof. □

This theorem gives a new family of bipartite distillable states under LOCC. In particular, for $M = N$ we obtain a positive answer to an open problem proposed in [37].

IV. REDUCIBLE AND IRREDUCIBLE STATES

In the previous sections we have shown that the states violating LFRP or RFRP are distillable and discussed several applications of this result. In this and subsequent sections we would like to explore further the structure of states having both full-rank properties. This problem is important for two main reasons. First, most bipartite quantum states have LFRP and RFRP, i.e., this is a generic property of the state space (see section VIII Second, by Theorem 10, the distillability problem for bipartite states reduces to the case of $M \times N$ states of rank bigger than $\max(M, N)$. We shall introduce some new concepts such as reducible states, and use them to construct new classes of distillable states. They complement Theorems 8 and 10.

We begin with the definition of reducible and irreducible states.

Definition 11 *The sum ρ of bipartite states ρ_i is* B-direct *if $\mathcal{R}(\rho_B)$ is a direct sum of $\mathcal{R}((\rho_i)_B)$. A bipartite state ρ is* reducible *if it is a B-direct sum of two states and otherwise ρ is* irreducible. *We denote them by ρ_{re} and ρ_{ir}, respectively.*

It is clear that if $A \otimes B$ is an arbitrary ILO, then ρ is reducible if and only if $(A \otimes B)\ \rho\ (A \otimes B)^\dagger$ is reducible.

We observe that in the case $N = 3$ the reducible NPT states are distillable.

Lemma 12 *Any $M \times 3$ reducible NPT state is distillable.*

Proof. Such a state ρ is a B-direct sum of two states with B-local ranks 1 and 2. Since the one with B-local rank 1 is separable, the other one must be NPT and so it is distillable by result (A). Consequently, ρ is distillable too. □

The following lemma is obvious.

Lemma 13 *Any bipartite state is a finite B-direct sum of irreducible states.*

We point out that this decomposition into irreducibles is not unique in general. For instance any 1×2 state ρ has infinitely many B-direct decompositions $\rho = \rho_1 + \rho_2$, where ρ_1 and ρ_2 are product states. Here is another more interesting example with entangled irreducible summands.

Example 14 Let ρ be the 4×4 (non-normalized) state which is by definition the B-direct sum $\rho = \rho_1 + \rho_2$ of two irreducible states $\rho_1 = 2|\phi_1\rangle\langle\phi_1|$ and $\rho_2 = 2|\phi_2\rangle\langle\phi_2|$ where $\phi_1 = |11\rangle + |22\rangle$ and $\phi_2 = |13\rangle + |24\rangle$. It admits another such decomposition $\rho = \rho'_1 + \rho'_2$, where $\rho'_1 = |\phi'_1\rangle\langle\phi'_1|$ and $\rho'_2 = |\phi'_2\rangle\langle\phi'_2|$ with

$$\phi'_1 = |11\rangle + |13\rangle + |22\rangle + |24\rangle \quad \text{and} \quad \phi'_2 = |11\rangle - |13\rangle + |22\rangle - |24\rangle.$$

The reducible states have the following important property.

Proposition 15 *Let $\rho = \rho_1 + \rho_2$ (a B-direct sum) be a reducible $M \times N$ state. Then there exists a Hermitian operator $V > 0$ on $\mathcal{H}_B$ such that the states $\rho'_i = (I \otimes V)\ \rho_i\ (I \otimes V)$ $(i = 1, 2)$ have orthogonal ranges, i.e., $(\rho'_1)_B(\rho'_2)_B = 0$.*

Proof. For convenience, set $\sigma_i = (\rho_i)_B$. Since $\mathcal{H}_B = \mathcal{R}(\sigma_1) \oplus \mathcal{R}(\sigma_2)$, we have $\sigma := \sigma_1 + \sigma_2 > 0$. We set $V = \sigma^{-1/2}$ and $\sigma_i' = V\sigma_i V$. Since $\sigma_1' + \sigma_2' = I_N$ and $\mathrm{rank}(I_N - \sigma_1') = \mathrm{rank}(\sigma_2') = N - \mathrm{rank}(\sigma_1')$, it follows that σ_1' is an orthogonal projector. Hence $\sigma_1'\sigma_2' = \sigma_1'(I_N - \sigma_1') = 0$. For $\rho_i' = (I \otimes V)\ \rho_i\ (I \otimes V)$, we have $(\rho_i')_B = V(\rho_i)_B V = V\sigma_i V = \sigma_i'$ and so $(\rho_1')_B(\rho_2')_B = \sigma_1'\sigma_2' = 0$. □

Corollary 16 *If $\rho = \sum_i \rho_i$ is a B-direct sum, then ρ is separable (PPT) if and only if every ρ_i is separable (PPT). In particular, the reducible 3×3 state ρ is separable if and only if it is PPT.*

For convenience, we refer to the following states [34, Sec. IV] as label states

$$\rho_{la} := \sum_i p_i |a_i, b_i\rangle\langle a_i, b_i|_{A_1B_1} \otimes |\psi_i\rangle\langle\psi_i|_{A_2B_2},$$

where the product states $|a_i, b_i\rangle$ are distinguishable via LOCC. In particular, when $|a_i, b_i\rangle = |1, i\rangle$, the state ρ_{la} becomes a reducible state. It is also known that three fundamental entanglement measures, i.e., the distillable entanglement E_d, entanglement cost E_c and entanglement of formation E_f are all equal for such states [34]. Indeed, because $|a_i, b_i\rangle$ are distinguishable via LOCC, by measurements Alice and Bob can get each pure state $|\psi_i\rangle$ with probability p_i. Therefore $E_d(\rho_{la}) = \sum_i p_i S(\mathrm{Tr}_A\, |\psi_i\rangle\langle\psi_i|)$, where $S(\rho)$ is the von Neumann entropy. On the other hand by the definition of E_c and E_f [9], we have the inequalities

$$E_d(\rho_{la}) \le E_c(\rho_{la}) \le E_f(\rho_{la}) \le \sum_i p_i S(\mathrm{Tr}_A |\psi_i\rangle\langle\psi_i|).$$

So the three entanglement measures coincide.

One can prove a similar fact for reducible states $\rho = \sum_i p_i \rho_i$, provided that $(\rho_i)_B(\rho_j)_B = 0$ for $i \ne j$. We assume here that the state ρ and the ρ_i are all normalized, $p_i > 0$, $\sum_i p_i = 1$, and the sum is B-direct. Thus Alice and Bob can get each state ρ_i with probability p_i. We have

$$E_d(\rho_{re}) = \sum_i p_i E_d(\rho_i) \le E_c(\rho_{re}) \le E_f(\rho_{re}) \le \sum_i p_i E_f(\rho_i). \tag{7}$$

Evidently, the three entanglement measures coincide if and only if $E_d(\rho_i) = E_f(\rho_i)$ for all i, e.g., this is satisfied for the label states [53]. Thus we have the following result.

Proposition 17 *If $\rho = \sum_i \rho_i$ is a B-direct sum, then ρ is distillable if and only if at least one of the ρ_i is distillable.*

Proof. By Proposition 15, we may assume that the states ρ_i are distinguishable by LOCC. From the equality in Eq. (7) we see that the inequality $E_d(\rho_{re}) > 0$ holds if and only if $E_d(\rho_i) > 0$ for at least one index i. □

Together with Lemma 13, this proposition shows that the question of deciding whether an arbitrary bipartite state is distillable reduces to the case of irreducible states. However, the distillability problem for irreducible states is hard. We tackle a special case in the following proposition.

Proposition 18 *Let ρ be an irreducible $M \times N$ state such that $\mathcal{H}_A' \otimes |b\rangle \subseteq \ker \rho$ for some $(M-1)$-dimensional subspace $\mathcal{H}_A' \subseteq \mathcal{H}_A$ and some state $|b\rangle \in \mathcal{H}_B$. Then ρ is distillable.*

Proof. We may assume that $\mathcal{H}_A'$ is spanned by the basis vectors $|i\rangle_A$, $i > 1$, and that $|b\rangle = |1\rangle_B$. By Proposition 6, we have $\rho = (C_1, \dots, C_M)^\dagger \cdot (C_1, \dots, C_M)$, where the C_i are $R \times N$ matrices and $R = \mathrm{rank}\, \rho$. Moreover, the first columns of the C_i are 0 for $i > 1$. Since ρ_B is invertible, the first column of C_1 is not 0. By multiplying $(C_1, \dots, C_M)$ by a unitary matrix on the left hand side, we may assume that only the first component of the first column of C_1 is nonzero. Clearly, we can also assume that the same is true for the first row of C_1. Since ρ is irreducible, at least one of the first rows of the C_i, $i > 1$, must be nonzero. It follows that ρ is trivially distillable. □

In view of Lemma 12, we have the following corollary.

Corollary 19 *If $N = 3$ we can remove the irreducibility hypothesis from Proposition 18.*

It is easy to construct examples of bipartite states having LFRP and RFRP which can be shown to be distillable by Proposition 18. Clearly, Theorem 8 cannot be applied to such states.

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[53] However, the equality does not hold for most of the known mixed bipartite states. The reason is that, according to the computable results, the distillable entanglement is smaller than the entanglement cost. It is also an important open problem to decide whether the equality holds only for the label states (including the pure states).

V. SEPARABILITY CRITERION FOR BIPARTITE STATES OF RANK 4

The problem of deciding whether a state is separable has been shown to be NP-hard [25], and hence the progress in solving instances of this problem is significant for both the quantum information theory and computer science. Most of the recent contributions focus on the numerical methods with objective to improve the efficiency in some special cases [27].

On the other hand the PPT condition, a celebrated decision criterion, is a necessary condition for separability of arbitrary $M \times N$ states. It is sufficient only if $MN \leq 6$. It is also sufficient for some other classes of bipartite states. For instance, this is the case when the rank, R, of ρ satisfies the inequality $R \leq \max(M,N)$. However, note that if $R < \max(M,N)$ then ρ is NPT by result (C). Thus, the PPT condition forces the inequality $\max(M,N) \leq R$. If $\max(M,N) = R$, then the above result applies and it remains to consider only the case $\max(M,N) < R$. We infer that if $R \leq 3$ then ρ is separable if and only if it is PPT, i.e., there are no PPT entangled states with $R \leq 3$. However, when $R = 4$ and $M = N = 3$ such states do exist and the problem arises to decide whether a given 3×3 PPT state of rank 4 is separable. In this section we provide a simple answer to this question, and thereby obtain a criterion for separability of arbitrary bipartite states of rank 4.

We start with an easy observation.

Lemma 20 *Let ρ be a $3 \times N$ state such that, for some $|a\rangle \in \mathcal{H}_A$, $\operatorname{rank}\langle a|\rho|a\rangle = 1$. If ρ is NPT then it is distillable. If ρ is PPT and $N = 3$, then ρ is separable.*

Proof. We shall prove both assertions at the same time. We may assume that $|a\rangle = |1\rangle$. Consequently, we have $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$, where the blocks C_i are $R \times N$, R is the rank of ρ, and C_1 has rank 1. After multiplying (C_1, C_2, C_3) by a unitary matrix on the left hand side and multiplying each C_i by the same invertible matrix on the right hand side, we may assume that

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} x & u \\ C_{21} & C_{22} \end{pmatrix}, \quad C_3 = \begin{pmatrix} y & v \\ C_{31} & C_{32} \end{pmatrix}, \tag{8}$$

where the first blocks are 1×1. If $u \neq 0$ or $v \neq 0$ then ρ is trivially distillable and so NPT. Thus we may assume that $u = v = 0$. By applying an ILO on $\mathcal{H}_A$, we may also assume that $x = y = 0$. After switching the parties A and B, ρ becomes reducible. Hence, if ρ is NPT it is distillable by Lemma 12, and otherwise it is separable by Corollary 16. $\square$

The following proposition is the crucial step in the proof of our separability criterion.

Proposition 21 *Let ρ be a 3×3 state of rank 4 containing at least one product state in its range. Then ρ is either distillable or separable. (In the former case it is NPT and in the latter PPT.) Equivalently, ρ cannot be PPT and entangled.*

Proof. If ρ is reducible, then the assertion follows from Lemma 12 and Corollary 16. Hence, we may assume that ρ is irreducible.

We can write ρ as $\rho = \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|$, where $|\psi_1\rangle$ a product state. This gives the factorization $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$ with the blocks C_i of size 4×3. By applying an ILO, we may assume that

$$C_1 = \begin{pmatrix} 1 & 0 \\ C_{11} & C_{12} \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ C_{21} & C_{22} \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ C_{31} & C_{32} \end{pmatrix},$$

where the blocks C_{i2} are of size 3×2. If the projected state $\rho' := (C_{12}, C_{22}, C_{32})^\dagger \cdot (C_{12}, C_{22}, C_{32})$ is entangled, then both ρ' and ρ are distillable. So, we may assume that ρ' is separable. Since a separable state of rank 3 is a sum of 3 product states (see Proposition 3), we may assume that $C_{i2} = D_i C$, $i = 1, 2, 3$, where C is a 3×2 matrix and the D_i are diagonal matrices. If D_2 and D_3 are linearly dependent, then we can assume that one of them is 0 and so Lemma 20 implies that ρ is distillable or separable. Thus we may assume that D_2 and D_3 are linearly independent. They remain independent after removing one of the rows in each of them, say the first row. By using an ILO on system A, we may assume that the diagonal entries of D_1, D_2 and D_3 are $(d_1, 0, 0)$, $(d_2, 1, 0)$ and $(d_3, 0, 1)$, respectively. The matrices C_i now have the form

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & d_1\alpha & d_1\beta \\ u_2 & 0 & 0 \\ u_3 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & d_2\alpha & d_2\beta \\ v_2 & \gamma & \delta \\ v_3 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ w_1 & d_3\alpha & d_3\beta \\ w_2 & 0 & 0 \\ w_3 & \epsilon & \zeta \end{pmatrix}.$$

If $d_1 = 0$ or $\alpha = \beta = 0$, we can again use Lemma 20. Thus, by using an ILO on system B, we may assume that $d_1 = \alpha = 1$ and $\beta = 0$, as well as that $u_1 = 0$. One of δ and ζ must be nonzero, and so we may assume that, say $\delta = 1$, as well as $v_2 = \gamma = 0$.

For convenience, denote by ρ_{ij} the projected state $(C_i, C_j)^\dagger \cdot (C_i, C_j)$, $1 \le i < j \le 3$. If $u_2 \neq 0$ then ρ_{12} is trivially distillable, and so we may assume that $u_2 = 0$. Now we have

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ u_3 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ v_1 & d_2 & 0 \\ 0 & 0 & 1 \\ v_3 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 0 \\ w_1 & d_3 & 0 \\ w_2 & 0 & 0 \\ w_3 & \epsilon & \zeta \end{pmatrix}.$$

By subtracting from C_2 and C_3 suitable multiples of C_1, we may assume that

$$C_2 = \begin{pmatrix} -d_2 & 0 & 0 \\ v_1 & 0 & 0 \\ 0 & 0 & 1 \\ v_3' & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -d_3 & 0 & 0 \\ w_1 & 0 & 0 \\ w_2 & 0 & 0 \\ w_3' & \epsilon & \zeta \end{pmatrix}.$$

If $v_1 \neq 0$ then ρ_{12} is trivially distillable, so we may assume that $v_1 = 0$.

Assume that $\zeta = 0$. Since ρ is irreducible, we must have $w_2 \neq 0$ and so ρ_{23} is trivially distillable. Assume now that $\zeta \neq 0$. If $u_3 \neq 0$ then ρ_{13} is trivially distillable, so we may assume that $u_3 = 0$. If $w_1 \neq 0$ then it is easy to see that the state ρ_{13} is distillable. Indeed, for that purpose we may assume that $\epsilon = 0$ and then ρ_{13} becomes trivially distillable. Thus we may assume that also $w_1 = 0$, and so

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -d_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ v_3' & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} -d_3 & 0 & 0 \\ 0 & 0 & 0 \\ w_2 & 0 & 0 \\ w_3' & \epsilon & \zeta \end{pmatrix}.$$

Since ρ is irreducible, we must have $\epsilon \neq 0$. If $v_3' \neq 0$ then ρ_{23} is trivially distillable. Thus we assume that $v_3' = 0$. If $w_2 = 0$ then ρ is separable. Thus we assume that $w_2 \neq 0$. Then we examine the projected state ρ_{23}. As $\epsilon \neq 0$, we can kill w_3' and ζ in C_3 to obtain a trivially distillable state. This completes the proof. □

Note that the above proof actually shows that the NPT states treated in this proposition are in fact 1-distillable.

We now present our separability criterion for bipartite states of rank 4.

Theorem 22 *A bipartite state of rank 4 is separable if and only if it is PPT and its range contains at least one product state.*

Proof. The conditions are obviously necessary. To prove the sufficiency, let ρ be an $M \times N$ bipartite state of rank 4. By the result (C), we must have $M, N \le 4$. If $\max(M, N) = 4$ then Proposition 3 shows that ρ is separable. In view of the Peres-Horodecki criterion, it remains to consider the case $M = N = 3$. Hence, we can invoke Proposition 21 to complete the proof. □

As an open question we ask whether Proposition 21 can be extended to some other classes of PPT states, such as 2×4 PPT entangled states of rank 5. Such states do exist [30] (see also [2, Eq. (6)]).

Hence to decide whether a 3×3 PPT state ρ of rank 4 is separable, one just has to check whether there is a product state in the range of ρ. This is numerically operational because the dimension is low. For example, one can refer to the discussion in [36, Sec. IV]. Here we present the analytical approach to this problem. We only need to consider the linear combination of four linearly independent 3×3 matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, $D = [d_{ij}]$. The problem is to decide whether the variables w, x, y, z can be chosen so that the matrix

$$E = wA + xB + yC + zD$$

has rank 1. This is equivalent to the requirement that each 2×2 minor of E vanishes. Mathematically, we have a set of nine equations

$$\begin{aligned} &(wa_{ij} + xb_{ij} + yc_{ij} + zd_{ij})(wa_{kl} + xb_{kl} + yc_{kl} + zd_{kl}) \\ &-(wa_{il} + xb_{il} + yc_{il} + zd_{il})(wa_{kj} + xb_{kj} + yc_{kj} + zd_{kj}) = 0, \end{aligned} \tag{9}$$

with $i < k$ and $j < l$. Thus $\operatorname{rank} E = 1$ if and only if these equations have a nonzero solution for w, x, y, z.

It is instructive to take a look at the following example which requires more algebraic background.

Example 23 Let us consider the simpler problem of deciding whether a 2-dimensional subspace V of a $2 \otimes 3$ system contains a product vector. Let $\{|a\rangle, |b\rangle\}$ be an arbitrary basis of V. Thus

$$|a\rangle = \sum_{i,j} a_{ij} |i,j\rangle, \quad |b\rangle = \sum_{i,j} b_{ij} |i,j\rangle,$$

where $i = 1, 2$ and $j = 1, 2, 3$. Let us form the 2×6 matrix from the components of these two states:

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \\ b_{11} & b_{12} & b_{13} & b_{21} & b_{22} & b_{23} \end{pmatrix}.$$

Denote by P^{ij}, $i < j$, the 2×2 submatrix of P made up from the ith and jth columns. The Plücker coordinates of V are the 15 determinants $p_{ij} = \det P^{ij}$. If we change the basis, the Plücker coordinates of V will be changed only by an overall factor. We point out that the Plücker coordinates are algebraically dependent. For instance, we have $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. A 2-dimensional subspace V can be viewed as a point of the so called Grassmann variety (or Grassmannian) $G_{2,6}$. The set of the points V which contain a product vector form a closed subvariety of this Grassmannian. By using the known facts about the incidence varieties as presented in [26], one can show that this subvariety is in fact a hypersurface, i.e., it is given by just one algebraic equation in the Plücker coordinates. This equation can be computed explicitly, it is given by a homogeneous polynomial of degree 3:

$$\begin{aligned} &2p_{12}p_{34}p_{56} + p_{12}p_{26}p_{46} + p_{13}p_{15}p_{56} + p_{23}p_{24}p_{46} + p_{13}p_{35}p_{45} \\ &-p_{13}p_{25}p_{46} - p_{13}p_{24}p_{56} - p_{12}p_{35}p_{46} - p_{12}p_{16}p_{56} - p_{23}p_{34}p_{45} = 0. \end{aligned}$$

Thus, V contains a product vector if and only if its Plücker coordinates satisfy this equation. The necessity can be easily verified by setting $a_{ij} = \alpha_i \beta_j$, computing the p_{ij}, and then verifying that the above equation is identically satisfied. For sufficiency, due to the fact that the p_{ij} are algebraically dependent, one must verify that the left hand side of the equation is not identically zero. For instance, if we set $a_{11} = a_{22} = b_{12} = b_{23} = 1$ and all other a_{ij} and b_{ij} set equal to 0, then the left hand side of the equation is equal to -1.

Similar polynomial equation exists for 3-dimensional subspaces V of a $2 \otimes 4$ system. The polynomial is again homogeneous, but now has degree 4, and it is an integer linear combination of 149 monomials in the Plücker coordinates p_{ijk}, $1 \le i < j < k \le 8$. This equation is given in the appendix. In the case of 4-dimensional subspaces of a $3 \otimes 3$ system again there is such an equation, but so far we were not able to compute it explicitly.

We note that Proposition 21 is in agreement with a conjecture proposed in [41], where the authors ask whether all 3×3 PPT entangled states of rank 4 are equivalent via stochastic LOCC to the PPT entangled states arising from UPB [8], i.e., the states $V_A \otimes V_B \ (I_9 - \sum_{i=1}^5 |a_i, b_i\rangle\langle a_i, b_i|) \ V_A^\dagger \otimes V_B^\dagger$ where V_A, V_B are invertible and $|a_i, b_i\rangle$ are the five members of a UPB. Indeed, Proposition 21 shows that $\mathcal{R}(\rho)$ contains no product state, and so provides evidence in support of the conjecture.

VI. SOME APPLICATIONS

In this section we apply Theorem 10 to study the quantum correlations inside tripartite pure states. We also discuss its meaning in terms of quantum discord [43]. We denote by d_A, d_B, d_C the dimensions of the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$, respectively.

As a bipartite state is always a reduced density operator of a tripartite pure state, say $\rho = |\psi\rangle\langle\psi|$, we can use Theorem 10 to characterize the quantum correlation by means of distillable entanglement. For example, we ask when are ρ_{AB} and ρ_{AC} simultaneously undistillable or bound entangled and which states $|\psi\rangle$ satisfy such a property. The question was first studied in 1999 by Thapliyal [48], who showed that an N-partite pure state $|\psi\rangle$ has fully $(N-1)$-partite separable states if and only if $|\psi\rangle$ is a GHZ state up to local unitary. Based on Theorem 10, we solve completely this problem for the tripartite system. We need a lemma to prove our theorem.

Lemma 24 *For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the bipartite reduced density operators ρ_{AB} and ρ_{AC} are PPT if and only if $|\psi\rangle = \sum_{i=1}^d |a_i\rangle|ii\rangle$ up to local unitary operations.*

Proof. The sufficiency is obvious, e.g., the reduced state $\rho_{AB} = \mathrm{Tr}_C\, \rho = \sum_i |a_i, i\rangle\langle a_i, i|$ is separable. Let us show the necessity. Suppose $\rho_{AB}^\Gamma \ge 0$ and $\rho_{AC}^\Gamma \ge 0$. Recall that $d_B = \operatorname{rank} \rho_{AC}$ and $d_C = \operatorname{rank} \rho_{AB}$. If follows from result (C) that $d_B \ge \max(d_A, d_C)$ and $d_C \ge \max(d_A, d_B)$. So we have $d := d_B = d_C \ge d_A$. According to

Proposition 3, the states ρ_{AB} and ρ_{AC} are separable and $\rho_{AB} = \sum_{i=1}^{d} |a_i, b_i\rangle\langle a_i, b_i|$, where the states $|b_i\rangle$ span $\mathcal{H}_B$. So, $|\psi\rangle = \sum_{i=1}^{d} |a_i, b_i, i\rangle$ up to local unitary operations.

We choose a subset $\{|g_1\rangle, \dots, |g_m\rangle\}$ of $\{|a_1\rangle, \dots, |a_d\rangle\}$ such that (i) $|g_i\rangle$ and $|g_j\rangle$ are not parallel if $i \neq j$ and (ii) each $|a_i\rangle$ is parallel to some $|g_j\rangle$. For each $k \in \{1, \dots, m\}$ let S_k be the set of all i such that $|a_i\rangle$ is parallel to $|g_k\rangle$. Thus the sets $S_1, \dots, S_m$ form a partition of the set $\{1, \dots, d\}$.

The state ρ_{AC} is also separable and has rank d. It follows from Proposition 3 again that $\rho_{AC} = \sum_{i=1}^{d} |e_i, f_i\rangle\langle e_i, f_i|$. Since the product states $|a_i, i\rangle$ span $\mathcal{R}(\rho_{AC})$, there is an invertible matrix $[c_{ij}]$ of order d such that

$$|e_i, f_i\rangle = \sum_{j=1}^{d} c_{ij} |a_j, j\rangle.$$

Since the $|j\rangle_C$ form an o.n. basis of $\mathcal{H}_C$, this equation implies that whenever coefficients c_{ij} and c_{il} are nonzero then $|a_j\rangle$ and $|a_l\rangle$ must be parallel. In other words there is a k such that $|e_i, f_i\rangle$ is a linear combination of $|g_k, j\rangle$ with $j \in S_k$. For each $k \in \{1, \dots, m\}$ let $\mathcal{H}_k$ be the subspace of $\mathcal{H}_C$ spanned by the $|j\rangle$ with $j \in S_k$, and let F_k be the set of all i such that $|e_i, f_i\rangle$ is a linear combination of $|g_k, j\rangle$ with $j \in S_k$. Clearly, the sets $F_1, \dots, F_m$ also form a partition of $\{1, \dots, d\}$. It is easy to see that, for each k, the sets F_k and S_k have the same cardinality, say d_k.

By using the spectral decomposition, we can rewrite the state $\sigma_k = \sum_{i \in F_k} |e_i, f_i\rangle\langle e_i, f_i|$ as $\sigma_k = \sum_{i=1}^{d_k} |g_k, f_{k,i}\rangle\langle g_k, f_{k,i}|$, where the states $|f_{k,i}\rangle$, $1 \leq i \leq d_k$, form an orthogonal basis of $\mathcal{H}_k$. By taking the sum of all σ_ks, and renaming all product states $|g_k, f_{k,i}\rangle$ as $|e'_j, f'_j\rangle$, we obtain

$$\rho_{AC} = \sum_{j=1}^{d} |e'_j, f'_j\rangle\langle e'_j, f'_j|,$$

where the states $|f'_j\rangle$ are pairwise orthogonal. The purification of this state, namely $|\psi\rangle$, reads $|\psi\rangle = \sum_{i=1}^{d} |e'_i\rangle|ii\rangle$, up to local unitary operations. This proves the necessity. □

Theorem 25 *For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the reduced states ρ_{AB} and ρ_{AC} are undistillable if and only if $|\psi\rangle = \sum_{i=1}^{d} |a_i\rangle|ii\rangle$ up to local unitary operations.*

Proof. The sufficiency readily follows from the separability of ρ_{AB} and ρ_{AC}. Let us show the necessity. As in the above proof we have $d_B = d_C \geq d_A$. By Theorem 10, the states ρ_{AB} and ρ_{AC} are PPT. Then the assertion follows from Lemma 24. □

Corollary 26 *For a tripartite state $|\psi\rangle$, all three bipartite reduced states are undistillable if and only if $|\psi\rangle$ is a generalized GHZ state, i.e., $|\psi\rangle = \sum_{i=1}^{d} a_i |iii\rangle$ up to local unitary operations.*

Theorem 25 reveals a new constraint for the distillable entanglement: When a tripartite pure state has two undistillable reduced density operators, then they have to be separable. In other words, when a tripartite pure state has a bound entangled reduced state, such as a PPT entangled state, then the other two reduced states must be NPT and distillable. So regardless of whether bound entanglement is PPT, or NPT as conjectured in [21], it is not a generic property shared by two parties of a tripartite pure state. The question whether a mixed tripartite state may have two bound entangled reduced states is still an open problem.

It is interesting that there is no counterpart to Corollary 26 for mixed states. For example, we consider the three-qubit PPT entangled state $\rho = I - \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|$, where the normalized $2 \otimes 2 \otimes 2$ unextendible product basis (UPB) reads [8, 10]

$$\begin{aligned}
|\psi_1\rangle &= |1, 1, 1\rangle, \\
|\psi_2\rangle &= |2, b, c\rangle, \\
|\psi_3\rangle &= |a, 2, c^{\perp}\rangle, \\
|\psi_4\rangle &= |a^{\perp}, b^{\perp}, 2\rangle,
\end{aligned} \tag{10}$$

and $|a\rangle, |a^{\perp}\rangle, |b\rangle, |b^{\perp}\rangle$ and $|c\rangle, |c^{\perp}\rangle$ are orthornormal, respectively. All three bipartitions of this state, i.e., $\rho_{A:BC}, \rho_{B:AC}, \rho_{C:AB}$ are separable. Hence all three bipartite reduced states are separable too. However the state ρ is PPT entangled and we cannot distill any (bipartite) pure entanglement for quantum information tasks. This is essentially different from Corollary 26. From sufficiently many copies of generalized GHZ state, one can asymptotically generate a standard GHZ state $|000\rangle + |111\rangle$ based on the BBPS protocol [4]. Therefore besides the distillable entanglement, we need additional parameters to characterize the quantum correlation inside mixed tripartite states.

On the other hand Theorem 25 can be better understood via quantum discord [43], which measures the bipartite quantum correlation beyond entanglement. Indeed, quantum discord is larger than zero for many separable states, and it is equal to zero iff the separable state is diagonal in one of the systems, e.g., $\sum_i \rho_i \otimes |i\rangle\langle i|$. In this state any measurement on system A will lead to pairwise commuting Hermitian operators on system B, which is a basic property of classical mechanics. In this sense, the system B is *classical* and it has no quantum features. Furthermore the converse statement is also true. That is, the state has the form $\sum_i \rho_i \otimes |i\rangle\langle i|$ if the system B is classical [13]. For simplicity we say that the state ρ_{AB} is classical when the system B is classical. By using Theorem 25 and the definition of quantum discord, we have

Lemma 27 *For a tripartite pure state $|\psi\rangle$, the reduced states ρ_{AB} and ρ_{AC} are undistillable if and only if they have zero quantum discord. In particular, $|\psi\rangle$ is a generalized GHZ state if and only if all reduced states have zero quantum discord.*

Lemma 27 shows that the collective undistillability of reduced states in a tripartite pure state will lead to the disappearance of quantum correlation among them. It is unknown whether this is also true for mixed tripartite states.

To grasp the meaning of this result more intuitively, we consider a practical case in which Alice and Bob are correlated in a state ρ_{AB}, and there is a third party, Charlie, from the environment. Suppose they are in a pure state $|\psi\rangle$ and share sufficiently many copies of $|\psi\rangle$. First, when at most one system is classical, say Alice, we can distill at least two bipartite reduced states by Theorem 25 and Lemma 27. Second, when two systems are classical, say Bob and Charlie, then there may be either one or two distillable bipartite reduced states. The former tripartite state is just $|\psi\rangle = \sum_{i=1}^{d} |a_i\rangle|ii\rangle$ of Lemma 27, while the latter corresponds to the purification of a fully classical state of the form $\sum_{ij} a_{ij}|ij\rangle\langle ij|$ up to local unitary [13]. In other words, the state $\sum_{ij} \sqrt{a_{ij}}|\varphi_{ij}\rangle|ij\rangle$ has classical systems B, C in the same reduced state. Meanwhile, we have distillable ρ_{AB} and ρ_{AC} when $d_A > \max(d_B, d_C)$ in view of result (C). Finally, when all three systems are classical then the state $|\psi\rangle$ is a generalized GHZ state. One cannot distill entanglement between any two parties. This indicates that the less classical systems there are in a composite system, the more distillable reduced states there are. This provides a connection between quantum discord and distillable entanglement, where the former is viewed as a kind of quantum correlation rather than entanglement measure [54].

VII. DISTILLATION OF SOME 3×3 STATES OF RANK 4

So far we have identified some distillable entangled states, e.g., Theorem 10 shows that $M \times N$ NPT states of rank N are 1-distillable. A natural question then arises: can we construct some more complex distillable states with specified dimension and rank? In this section we focus on the simplest nontrivial case, namely 3×3 states of rank 4. In particular we will show that the checkerboard states [23], which generalize the celebrated Bruß-Peres family, are distillable if and only if they are NPT. Some of the results are also extendible to higher dimensions.

We shall now examine the so called *checkerboard states*, i.e., the states ρ acting on a $3 \otimes 3$ system and defined by Eq. (3) with $M = N = 3$, $R = 4$, and the $|\psi_i\rangle$ given by:

$$
\begin{aligned}
|\psi_1\rangle &= |1\rangle(a|1\rangle + d|3\rangle) + |2\rangle(c|2\rangle) + |3\rangle(b|1\rangle + e|3\rangle), \\
|\psi_2\rangle &= |1\rangle(g|2\rangle) + |2\rangle(f|1\rangle + i|3\rangle) + |3\rangle(h|2\rangle), \\
|\psi_3\rangle &= |1\rangle(j|1\rangle + m|3\rangle) + |2\rangle(l|2\rangle) + |3\rangle(k|1\rangle + n|3\rangle), \\
|\psi_4\rangle &= |1\rangle(q|2\rangle) + |2\rangle(p|1\rangle + s|3\rangle) + |3\rangle(r|2\rangle).
\end{aligned}
$$

The parameters $a, b, \ldots, s$ are arbitrary complex numbers. Each pure state $|\psi_i\rangle$ is written above in the form $|\psi_i\rangle = \sum_j |i\rangle|\psi_{ij}\rangle$. By using these $|\psi_{ij}\rangle$ and Eq. (4) we have $\rho = (C_1, C_2, C_3)^\dagger \cdot (C_1, C_2, C_3)$, where the complex conjugates of the blocks C_i are given by:

$$
C_1^* = \begin{pmatrix} a & 0 & d \\ 0 & g & 0 \\ j & 0 & m \\ 0 & q & 0 \end{pmatrix}, \quad
C_2^* = \begin{pmatrix} 0 & c & 0 \\ f & 0 & i \\ 0 & l & 0 \\ p & 0 & s \end{pmatrix}, \quad
C_3^* = \begin{pmatrix} b & 0 & e \\ 0 & h & 0 \\ k & 0 & n \\ 0 & r & 0 \end{pmatrix}.
$$

[54] A connection between quantum discord and the entanglement of formation, which is another important entanglement measure, has been found by several authors, see e.g., quant-ph/1007.1814, quant-ph/1006.4727.

Generically, these states ρ have rank 4 and $\mathcal{R}(\rho)$ contains no product state (see [23]) and, consequently, ρ is entangled. One can find in the same paper two concrete examples of NPT checkerboard states ρ which are distillable. In the next theorem we prove that this feature is shared by all NPT checkerboard states.

Theorem 28 *All NPT checkerboard states are distillable.*

Proof. Let ρ be an NPT checkerboard state. We have to show that ρ is distillable. This is certainly the case if one of the local ranks of ρ is less than 3. Hence we may assume that ρ is a 3×3 state. We divide the proof into two steps. First, we eliminate all but five parameters from the matrices C_i^*. Second, we analytically investigate the partial transpose of various 2×2 and 2×3 projected states of ρ. If any of them is entangled then ρ is distillable and so we can dismiss such cases.

Let us carry out the first step. By an argument similar to one in the beginning of the proof of Proposition 21, we can assume that $b = e = j = m = 0$. By using an ILO, we can also assume that $a = g = 1$ and $d = q = 0$. Thus we have

$$C_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2^* = \begin{pmatrix} 0 & c & 0 \\ f & 0 & i \\ 0 & l & 0 \\ p & 0 & s \end{pmatrix}, \quad C_3^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ k & 0 & n \\ 0 & r & 0 \end{pmatrix}.$$

If $i \neq 0$ then ρ is trivially distillable, and so we assume that $i = 0$. If $s = 0$ then ρ is distillable by Corollary 19, so we may assume that $s \neq 0$. Similarly, we may assume that $n \neq 0$. By multiplying C_2^* by $1/f$, we may assume that $f = 1$. By multiplying the third columns of C_i^* by $1/s$, we may also assume that $s = 1$. By multiplying C_3^* by $1/n$, we may assume that $n = 1$. By using an ILO we can also assume that $p = 0$. We now have

$$C_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2^* = \begin{pmatrix} 0 & c & 0 \\ 1 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_3^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & 0 \\ k & 0 & 1 \\ 0 & r & 0 \end{pmatrix}.$$

This completes the first step.

Now we carry out the second step. Let $\sigma = (V \otimes I)\ \rho\ (V^\dagger \otimes I)$, where $V^\dagger = (x|1\rangle + |3\rangle)\langle 1| + |2\rangle\langle 2|$ and x is a complex parameter. Assume that σ is PPT for all x. After permuting simultaneously the rows and columns of σ^Γ, we obtain a direct sum $A \oplus B$ of two 3×3 Hermitian matrices. Since σ is PPT, we must have

$$\begin{aligned} \det A &= |x|^2(|c|^2 + |l|^2 - |r|^2) - |x + h^* - kr^*|^2 \geq 0, \\ \det B &= |x + h^*|^2 - |cx + lk^*|^2 + |r|^2 - |l|^2 \geq 0. \end{aligned}$$

By setting $x = 0$, the first inequality gives $h = rk^*$. After eliminating h, the above inequalities imply that $|c| = 1$ and $|l| = |r|$, and finally that $l = cr^*k/k^*$ if $k \neq 0$. By using these equalities, a computation shows that ρ is PPT which is a contradiction. Hence σ must be NPT for some x, and so this particular σ and ρ are distillable. One can similarly and more easily handle the case $k = 0$. This completes the proof. □

VIII. MORE ON FULL-RANK PROPERTIES

Theorem 8 gives a new theoretical tool, the full-rank criterion, for detection of bipartite states distillable under LOCC. This criterion is similar to the well-known reduction criterion [28], for both criteria ensure the distillability of the states which violate them. It is easy to show that these two criteria are incomparable. First, the distillable 3×3 antisymmetric Werner state ρ_{as} is detected by the full-rank criterion but not by reduction criterion. Second, there are 1-distillable states ρ which can be detected by reduction criterion but not by the full-rank criterion. An example is the 2×2 entangled state $\rho = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$, where $|\psi_1\rangle = \sqrt{p/2}(|00\rangle + |11\rangle)$ and $|\psi_2\rangle = \sqrt{(1-p)/2}(|00\rangle - |11\rangle)$ with $0 < p < 1$ and $p \neq 1/2$. It follows from Proposition 30 below that all 2×2 states have both LFRP and RFRP. Third, some distillable states are detected by both criteria such as $M \times N$ states of rank $< N$; in particular the pure entangled states.

Since the states violating either criterion are distillable, they must be NPT. In other words, the PPT states satisfy both reduction criterion and the full-rank criterion.

Next we consider the states with RFRP. Let ρ be an $M \times N$ state of rank R written as in Eq. (2) where the C_i are $R \times N$ matrices. The matrix of ρ is the block matrix $(C_i^\dagger C_j)$, $i, j = 1, \dots, M$, where each block is an $N \times N$ matrix.

If $\text{rank}(C_i) < N$, then the rank of the submatrix $\left(C_i^\dagger C_1, \ldots, C_i^\dagger C_M\right)$ is $< N$. Hence, if $\text{rank}(C_i) < N$ for all i, then $R \leq M(N-1)$. We conclude that if the rank of ρ exceeds $M(N-1)$ then ρ must have RFRP (in fact one of the C_is must have rank N).

On the other hand, if $R < N$ then ρ violates RFRP. This follows from the fact that $\text{rank}(\langle x|\rho|x\rangle) \leq R$ for all $|x\rangle \in \mathcal{H}_A$. Moreover, the $N \times N$ states ρ in the antisymmetric space violate RFRP and some of them have rank as large as $N(N-1)/2$. To prove this, note that the states $|\pi_{ij}\rangle = |ij\rangle - |ji\rangle$, $1 \leq i < j \leq N$, form a basis of the antisymmetric space $\mathcal{A}$. We have $\rho = \sum_k |\psi_k\rangle\langle\psi_k|$, where $|\psi_k\rangle \in \mathcal{A}$ are non-normalized states. Let $|x\rangle \in \mathcal{H}_A$ be a nonzero vector. For $i < j$ we have

$$\langle \pi_{ij}|x, x\rangle = \langle i, j|x, x\rangle - \langle j, i|x, x\rangle = 0.$$

Since each $|\psi_k\rangle$ is a linear combination of the $|\pi_{ij}\rangle$, it follows that $|x, x\rangle \in \ker \rho$. Consequently, ρ violates RFRP.

For convenience, we collect the above results in a proposition.

Proposition 29 *Let R be the maximum rank of $M \times N$ states which violate RFRP. Then $R \leq M(N-1)$ and, if $M = N$, $R \geq N(N-1)/2$.*

How can we verify whether ρ has RFRP? To answer this question, let us write ρ as in Eq. (2), where the C_i are $R \times N$ matrices and $R = \text{rank}(\rho)$. For $|x\rangle_A = \sum_k \xi_k |k\rangle$, we have

$$\langle x|\rho|x\rangle = \left(\sum_k \xi_k C_k\right)^\dagger \cdot \left(\sum_k \xi_k C_k\right),$$

and so $\text{rank}(\langle x|\rho|x\rangle) = \text{rank}(\sum_k \xi_k C_k)$. For small N the answer to our question can be obtained simply by computing all $N \times N$ minors of the matrix $\sum_k \xi_k C_k$.

In general, ρ violates RFRP if and only if the space of $R \times N$ matrices spanned by $C_1, \ldots, C_M$ contains no matrix of rank N. This is certainly the case if $R < N$. The above problem is related to the still open problem of Edmonds [24], from theoretical computer science, which asks to decide whether a given linear subspace of complex $M \times N$ ($M \leq N$) matrices contains a matrix of rank M.

A similar test is valid for the LFRP. For $|y\rangle_B = \sum_l \eta_l |l\rangle$, we have $\langle y|\rho|y\rangle = \left(\langle y|C_i^\dagger C_j|y\rangle\right)$, where the right hand member is an $M \times M$ matrix with the indicated entries. It follows that the rank of $\langle y|\rho|y\rangle$ is equal to the rank of the $R \times M$ matrix $(C_1|y\rangle, \ldots, C_M|y\rangle)$. Let K_i, $i = 1, \ldots, N$, be the $R \times M$ matrix defined as follows: the jth column of K_i is the ith column of C_j. Then ρ violates LFRP if and only if the space of $R \times M$ matrices spanned by $K_1, \ldots, K_N$ contains no matrix of rank M.

It is not hard to construct examples of states which possess LFRP but violate RFRP. In the next proposition we consider the RFRP for $M \times 2$ states.

Proposition 30 *All $M \times 2$ states have RFRP. Equivalently, all $2 \times N$ states have LFRP.*

Proof. The equivalence is clear since we can interchange the parties A and B. We shall prove only the first assertion.

The proof is by contradiction. Let us assume that there exists an $M \times 2$ state ρ which violates RFRP. Then the rank, R, of ρ must be at least two. We have

$$\rho = \sum_{i,j=1}^{M} |i\rangle\langle j| \otimes C_i^\dagger C_j,$$

where C_i are $R \times 2$ matrices. As $\text{rank}(\rho_A) = M$, $C_i \neq 0$ for each index i. Since ρ violates RFRP, the matrix $\sum_i \xi_i C_i$ must have rank less than two for arbitrary choice of complex numbers ξ_i. It follows that each C_i must have rank one and so $C_i = |v_i\rangle\langle\phi_i|$, where $|\phi_i\rangle = \alpha_i|1\rangle + \beta_i|2\rangle \in \mathcal{H}_B$ and $|v_i\rangle$ are nonzero vectors.

We have $C_i^\dagger C_j = \langle v_i|v_j\rangle |\phi_i\rangle\langle\phi_j|$. Hence

$$\rho_B = \sum_i C_i^\dagger C_i = \sum_i \|v_i\|^2 \begin{pmatrix} |\alpha_i|^2 & \alpha_i \beta_i^* \\ \alpha_i^* \beta_i & |\beta_i|^2 \end{pmatrix}.$$

By using the Lagrange identity, we find that

$$\begin{aligned} \det(\rho_B) &= \left(\sum_i |\alpha_i|^2 \|v_i\|^2\right)\left(\sum_i |\beta_i|^2 \|v_i\|^2\right) - \left|\sum_i |\alpha_i \beta_i^*| \|v_i\|^2\right|^2 \\ &= \sum_{i<j} \|v_i\|^2 \|v_j\|^2 |\alpha_i \beta_j - \alpha_j \beta_i|^2. \end{aligned}$$

Since all $|v_i\rangle \neq 0$ and $\det(\rho_B) > 0$, there exists a pair of indexes i, j such that $\alpha_i \beta_j - \alpha_j \beta_i \neq 0$. Without any loss of generality we may assume that $i = 1$ and $j = 2$. We know that the $R \times 2$ matrix

$$\begin{aligned} \xi_1 C_1 + \xi_2 C_2 &= \xi_1 |v_1\rangle\langle\phi_1| + \xi_2 |v_2\rangle\langle\phi_2| \\ &= (\xi_1 \alpha_1^* |v_1\rangle + \xi_2 \alpha_2^* |v_2\rangle, \xi_1 \beta_1^* |v_1\rangle + \xi_2 \beta_2^* |v_2\rangle) \end{aligned}$$

has rank less than two for arbitrary complex numbers ξ_1 and ξ_2. Thus all of its 2×2 minors must vanish:

$$\begin{aligned} 0 &= \begin{vmatrix} \xi_1 \alpha_1^* v_{1,k} + \xi_2 \alpha_2^* v_{2,k} & \xi_1 \beta_1^* v_{1,k} + \xi_2 \beta_2^* v_{2,k} \\ \xi_1 \alpha_1^* v_{1,l} + \xi_2 \alpha_2^* v_{2,l} & \xi_1 \beta_1^* v_{1,l} + \xi_2 \beta_2^* v_{2,l} \end{vmatrix} \\ &= \xi_1 \xi_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1)^* (v_{1,k} v_{2,l} - v_{2,k} v_{1,l}), \end{aligned}$$

where $1 \leq k < l \leq M$ and $v_{i,k}$ denotes the kth component of $|v_i\rangle$. Since ξ_1 and ξ_2 are arbitrary and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, we have $v_{1,k} v_{2,l} - v_{2,k} v_{1,l} = 0$ for all $k < l$. Hence the vectors $|v_1\rangle$ and $|v_2\rangle$ are linearly dependent. Without any loss of generality we may assume that $|v_1\rangle = |v_2\rangle$.

Recall from Eqs. (3) and (4) that we can write

$$\rho = \sum_{i=1}^{R} |\psi_i\rangle\langle\psi_i|,$$

where $|\psi_i\rangle = \sum_j |j\rangle \otimes |\psi_{ij}\rangle$ and $\langle\psi_{ij}|$ is the ith row of C_j, i.e.,

$$|\psi_{ij}\rangle = v_{j,i}^* (\alpha_j |1\rangle + \beta_j |2\rangle) = v_{j,i}^* |\phi_j\rangle.$$

Since $|v_1\rangle = |v_2\rangle$ we obtain that

$$|\psi_i\rangle = v_{1,i}^* (|1, \phi_1\rangle + |2, \phi_2\rangle) + \sum_{j>2} v_{j,i}^* |j, \phi_j\rangle, \quad i = 1, \dots, R.$$

We conclude that $\mathcal{R}(\rho)$ is contained in the subspace spanned by the vectors $|1, \phi_1\rangle + |2, \phi_2\rangle$ and $|j, \phi_j\rangle$ for $j = 3, \dots, R$. This contradicts the hypothesis that $\operatorname{rank}(\rho) = R$, and completes the proof. □

In particular, all 2×2 states have LFRP and RFRP. While all $2 \times N$ states have LFRP, even the simplest non-trivial case, i.e., a 2×3 state, may violate RFRP. Indeed, one can easily verify that the 2×3 state $\rho = (|11\rangle + |22\rangle)(\langle 11| + \langle 22|) + |23\rangle\langle 23| + |13\rangle\langle 13|$ violates RFRP.

Hence for any $2 \times N$ state ρ of rank $R \leq N$ there exists a state $|x\rangle \in \mathcal{H}_A$ such that $\operatorname{rank}\langle x|\rho|x\rangle < N$. However this is not true for $R > N$.

Finally, we have seen that the antisymmetric state ρ_{as} given by Eq. (5) violates RFRP. However, as one can easily verify, its two-copy version $\rho_{as} \otimes \rho_{as}$ has RFRP.

IX. CONCLUSIONS

Following [36], we have introduced two full-rank properties, LFRP and RFRP, and proved that the bipartite state violating at least one of them is 1-distillable. We refer to this result as the full-rank criterion for distillability. By using it, we obtained our first main result which asserts that the $M \times N$ NPT states of rank N are 1-distillable. In particular, this result provides the affirmative solution to an open problem first proposed in 1999. This result leads to a new characterization of the distillable entanglement, namely a tripartite pure state $|\psi\rangle$ cannot have two undistillable entangled bipartite reduced states. We also derived an explicit expression for tripartite pure states having two undistillable bipartite reduced states. Both of these states turn out to be separable. On the other hand, we define reducible and irreducible states and use them to distill some entangled states possessing both LFRP and RFRP. The most important result in this direction is that the checkerboard states [23] are distillable under LOCC if and only if they are NPT.

We now list the most interesting results proved in this paper. We shall just continue the enumeration of the results (A-E) from the Introduction.

(F) Bipartite $M \times N$ NPT states of rank $\max(M, N)$ are 1-distillable.

(G) A bipartite rank-4 state is separable if and only if it is PPT and its range contains at least one product state.

(H) If a bipartite state violates at least one of the two full-rank conditions, then it is 1-distillable.

(I) For a tripartite pure state $|\psi\rangle$, all three bipartite reduced states are undistillable if and only if $|\psi\rangle$ is a generalized GHZ state.

(J) The NPT checkerboard states (acting on a $3 \otimes 3$ system) are 1-distillable.

For convenience, let us summarize what is known and what remains to be done to finish off the coarse classification of rank 4 states ρ in a $3 \otimes 3$ system. There are two cases:
1) $\mathcal{R}(\rho)$ contains a product state;
2) $\mathcal{R}(\rho)$ contains no product state.
In case 1), we know that if ρ is PPT then it is separable, and otherwise it is 1-distillable.
In case 2), ρ is always entangled and may be PPT or NPT. For the PPT subcase, there is a very precise conjecture [41] describing how these PPT entangled states can be generated from UPBs. For the NPT subcase, we conjecture that these states are distillable, and we know that this is true for the checkerboard states (see Theorem 28).

There are many related open problems and conjectures for further study that originate from this paper. First, we have seen from Proposition 18 that an irreducible bipartite state ρ, for which there exists a nonzero vector $|x\rangle$ such that $C_i|x\rangle = 0$ for all but one of the indexes i, is distillable. (Note that if $C_i|x\rangle = 0$ then $|i, x\rangle \in \ker \rho$.) It is thus natural to ask whether we can relax the above condition on the kernels of the C_is. For example one can conjecture that any irreducible $M \times N$ NPT state ρ, with one of the C_i of deficient rank, is distillable? Since Werner states have full rank, all of its blocks C_i also have full rank, and so they do not contradict the conjecture. Because of the rank condition on the C_i, the question evidently depends on the analysis of product states in the kernel of ρ. For example, it is a well-known fact that in an $M \otimes N$ system every subspace $V \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$ with $\operatorname{Dim} V > (M-1)(N-1)$ contains a product vector. To prove this, we can just apply [26, Proposition 11.4] to the Segre variety consisting of all product vectors.

If the above conjecture turns out to be true, it would follow that every bipartite $M \times N$ ($M \le N$) NPT state ρ of rank 4 is distillable. Indeed, if $M < 3$ then ρ is distillable by result (A). If $N > 4$ then ρ is distillable by result (C). Thus we may assume that $M = N = 3$. As $\ker \rho$ has dimension 5, it contains a product state. By the conjecture, ρ is distillable.

To state the second conjecture, let ρ be a bipartite NPT state acting on some $M \otimes N$ system. We can view $\rho^{\otimes k}$ as a bipartite state acting on a $M^k \otimes N^k$ system. Then the conjecture claims that for some $k \ge 1$, the state $\rho^{\otimes k}$ can be locally transformed into an $m \times n$ NPT state of rank $\max(m, n) + 1$.

If this conjecture is true, then we can project some tensor power of an entangled 1-undistillable Werner state ρ_w onto an $m \times n$ ($m \le n$) NPT state σ of rank $n + 1$. Because it is widely believed that ρ_w is undistillable, we can thus conjecture that there exists an undistillable $M \times N$ NPT state of rank $N + 1$. This would then imply that the value N for the rank of the states in Theorem 10 is maximal.

The third question is about the irreducible states. From Lemma 13 we know that the distillation problem relies on further investigation of irreducible states. For example we ask: can the tensor product of two irreducible states be a reducible state? The existence of such phenomenon would become a sort of activation of reducibility, which is akin to the activation of PPT bound entanglement [33].

We also have seen that all $M \times N$ NPT states of rank N are 1-distillable. In fact previously researchers have shown that such states cannot be PPT entangled. However it is also well-known that there are PPT entangled $M \times N$ states with rank bigger than N, and meanwhile, there exist 1-undistillable but n-distillable NPT states such as the Watrous state [51]. Is this an essential difference between the above two families of states? Does the existence of PPT entanglement imply that of 1-undistillable NPT states?

Another interesting problem is to decide which PPT states are entangled. For example, what is the relationship between the existence of product states in the range and entanglement of PPT states? Can we conjecture that if there are more known product states in the range of a PPT state, then it becomes easier to decide whether it is entangled?

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X. APPENDIX

We give here the equation which is necessary and sufficient for a 3-dimensional subspace V of a $2 \otimes 4$ system to contain at least one product state. Let $\{|a\rangle, |b\rangle, |c\rangle\}$ be an arbitrary basis of V. Thus

$$|a\rangle = \sum_{i=1}^{2} \sum_{j=1}^{4} a_{ij} |i, j\rangle,$$

and similarly for $|b\rangle$ and $|c\rangle$. Let us form the 3×8 matrix from the components of these three states:

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{21} & a_{22} & a_{23} & a_{24} \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{21} & b_{22} & b_{23} & b_{24} \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{21} & c_{22} & c_{23} & c_{24} \end{pmatrix}.$$

Denote by P^{ijk}, $1 \le i < j < k \le 8$, the 3×3 submatrix of P made up from the ith, jth and kth columns. The Plücker coordinates of V are the 56 determinants $p_{ijk} = \det P^{ijk}$. If we change the basis, the Plücker coordinates of V will be changed only by an overall factor. The equation we aluded to is the following:

$$p_{123}F_1 + p_{124}F_2 + p_{134}F_3 + p_{234}F_4 = 0,$$

where

$$\begin{aligned} F_1 = {} & 3p_{124}p_{578}p_{678} - 3p_{125}p_{478}p_{678} + p_{126}p_{478}p_{578} + 2p_{127}p_{458}p_{678} - p_{127}p_{468}p_{578} + p_{128}p_{178}p_{678} - p_{128}p_{278}p_{578} \\ & -2p_{128}p_{358}p_{678} + p_{128}p_{368}p_{578} - 3p_{134}p_{568}p_{678} + 3p_{135}p_{468}p_{678} - p_{136}p_{458}p_{678} - p_{136}p_{468}p_{578} + p_{137}p_{468}p_{568} \\ & -p_{138}p_{168}p_{678} + p_{138}p_{258}p_{678} + p_{138}p_{268}p_{578} - p_{138}p_{368}p_{568} + 2p_{145}p_{278}p_{678} - p_{145}p_{368}p_{678} - 2p_{145}p_{467}p_{678} \\ & -2p_{146}p_{178}p_{678} + 2p_{146}p_{278}p_{578} + 4p_{146}p_{358}p_{678} - 2p_{146}p_{368}p_{578} - 2p_{146}p_{457}p_{678} + 2p_{146}p_{467}p_{578} + p_{147}p_{168}p_{678} \\ & -p_{147}p_{258}p_{678} - p_{147}p_{268}p_{578} + p_{147}p_{368}p_{568} - p_{147}p_{467}p_{568} + 3p_{234}p_{568}p_{578} - 2p_{235}p_{458}p_{678} - 2p_{235}p_{468}p_{578} \\ & +2p_{236}p_{458}p_{578} - p_{237}p_{458}p_{568} - p_{238}p_{258}p_{578} + p_{238}p_{358}p_{568} - 2p_{245}p_{278}p_{578} - 2p_{245}p_{358}p_{678} + 4p_{245}p_{368}p_{578} \\ & +3p_{245}p_{457}p_{678} - p_{245}p_{467}p_{578} - p_{246}p_{358}p_{578} - p_{246}p_{457}p_{578} + p_{247}p_{258}p_{578} - p_{247}p_{358}p_{568} + p_{247}p_{457}p_{568} \\ & -2p_{345}p_{368}p_{568} - 3p_{345}p_{456}p_{678} + 2p_{345}p_{467}p_{568} + p_{346}p_{358}p_{568} + p_{346}p_{456}p_{578} - p_{346}p_{457}p_{568}, \end{aligned}$$

$$\begin{aligned} F_2 = {} & p_{125}p_{378}p_{678} - p_{127}p_{178}p_{678} + p_{127}p_{278}p_{578} + 2p_{127}p_{358}p_{678} - p_{127}p_{368}p_{578} - 2p_{127}p_{457}p_{678} + p_{127}p_{467}p_{578} \\ & +3p_{134}p_{567}p_{678} - p_{135}p_{278}p_{678} - p_{135}p_{368}p_{678} - 2p_{135}p_{467}p_{678} + p_{136}p_{178}p_{678} - p_{136}p_{278}p_{578} - 2p_{136}p_{358}p_{678} \\ & +p_{136}p_{368}p_{578} + 4p_{136}p_{457}p_{678} - p_{136}p_{467}p_{578} + p_{137}p_{168}p_{678} - p_{137}p_{258}p_{678} - p_{137}p_{268}p_{578} + p_{137}p_{368}p_{568} \\ & -p_{137}p_{467}p_{568} + 2p_{145}p_{367}p_{678} - p_{146}p_{357}p_{678} - p_{147}p_{167}p_{678} + p_{147}p_{257}p_{678} + p_{147}p_{267}p_{578} - p_{147}p_{367}p_{568} \\ & +p_{147}p_{467}p_{567} - 3p_{234}p_{567}p_{578} + p_{235}p_{278}p_{578} + 2p_{235}p_{358}p_{678} - p_{235}p_{368}p_{578} - p_{235}p_{457}p_{678} + 4p_{235}p_{467}p_{578} \\ & -2p_{236}p_{457}p_{578} + p_{237}p_{258}p_{578} - p_{237}p_{358}p_{568} + p_{237}p_{457}p_{568} - p_{245}p_{357}p_{678} - 2p_{245}p_{367}p_{578} + p_{246}p_{357}p_{578} \\ & -p_{247}p_{257}p_{578} + p_{247}p_{357}p_{568} - p_{247}p_{457}p_{567} + p_{345}p_{356}p_{678} + 2p_{345}p_{367}p_{568} - 2p_{345}p_{467}p_{567} - p_{346}p_{357}p_{568} \\ & +p_{346}p_{457}p_{567}, \end{aligned}$$

$$\begin{aligned} F_3 = {} & p_{135}p_{268}p_{678} - p_{136}p_{168}p_{678} + p_{136}p_{258}p_{678} + p_{136}p_{268}p_{578} - p_{136}p_{368}p_{568} - 2p_{136}p_{456}p_{678} + p_{136}p_{467}p_{568} \\ & -p_{145}p_{267}p_{678} + p_{146}p_{167}p_{678} - p_{146}p_{257}p_{678} - p_{146}p_{267}p_{578} + p_{146}p_{356}p_{678} + p_{146}p_{367}p_{568} - p_{146}p_{467}p_{567} \\ & -p_{235}p_{258}p_{678} - p_{235}p_{268}p_{578} + p_{235}p_{368}p_{568} + 2p_{235}p_{456}p_{678} - p_{235}p_{467}p_{568} - p_{236}p_{258}p_{578} + p_{236}p_{358}p_{568} \\ & +2p_{236}p_{456}p_{578} - p_{236}p_{457}p_{568} + p_{245}p_{257}p_{678} + p_{245}p_{267}p_{578} - p_{245}p_{356}p_{678} - p_{245}p_{367}p_{568} + p_{245}p_{467}p_{567} \\ & +p_{246}p_{257}p_{578} - p_{246}p_{356}p_{578} - p_{246}p_{357}p_{568} + p_{246}p_{457}p_{567} + p_{346}p_{356}p_{568} - p_{346}p_{456}p_{567}, \end{aligned}$$

$$\begin{aligned} F_4 = {} & p_{235}p_{258}p_{578} - p_{235}p_{358}p_{568} - 2p_{235}p_{456}p_{578} + p_{235}p_{457}p_{568} - p_{245}p_{257}p_{578} + p_{245}p_{356}p_{578} + p_{245}p_{357}p_{568} \\ & -p_{245}p_{457}p_{567} - p_{345}p_{356}p_{568} + p_{345}p_{456}p_{567}. \end{aligned}$$

Although this equation looks complicated, it can be easily programmed on a computer and used to decide whether V contains a product state. The main point is that we do not need to solve numerically any algebraic equations.